THEORY OF ELASTICITY

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THEORY

OF

ELASTICITY

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THEORY OF ELASTICITY

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PREFACE TO THE SECOND EDITION

The many developments and clarifications in the theory of elasticity and its applications which have occurred since the first edition was written are reflected in numerous additions and emendations in the present edition. The arrangement of the book remains the same for the most part.

The treatments of the photoelastic method, two-dimensional problems in curvilinear coordinates, and thermal stress have been rewritten and enlarged into separate new chapters which present many methods and solutions not given in the former edition. An appendix on the method of finite differences and its applications, including the relaxation method, has been added. New articles and paragraphs incorporated in the other chapters deal with the theory of the strain gauge rosette, gravity stresses, Saint-Venant's principle, the components of rotation, the reciprocal theorem, general solutions, the approximate character of the plane stress solutions, center of twist and center of shear, torsional stress concentration at fillets, the approximate treatment of slender (*e.g.*, solid airfoil) sections in torsion and bending, and the circular cylinder with a band of pressure.

Problems for the student have been added covering the text as far as the end of the chapter on torsion.

It is a pleasure to make grateful acknowledment of the many helpful suggestions which have been contributed by readers of the book.

S. Timoshenko
J. N. Goodier

Palo Alto, Calif.
February, 1951

PREFACE TO THE FIRST EDITION

During recent years the theory of elasticity has found considerable application in the solution of engineering problems. There are many cases in which the elementary methods of strength of materials are inadequate to furnish satisfactory information regarding stress distribution in engineering structures, and recourse must be made to the more powerful methods of the theory of elasticity. The elementary theory is insufficient to give information regarding local stresses near the loads and near the supports of beams. It fails also in the cases when the stress distribution in bodies, all the dimensions of which are of the same order, has to be investigated. The stresses in rollers and in balls of bearings can be found only by using the methods of the theory of elasticity. The elementary theory gives no means of investigating stresses in regions of sharp variation in cross section of beams or shafts. It is known that at reentrant corners a high stress concentration occurs and as a result of this cracks are likely to start at such corners, especially if the structure is submitted to a reversal of stresses. The majority of fractures of machine parts in service can be attributed to such cracks.

During recent years considerable progress has been made in solving such practically important problems. In cases where a rigorous solution cannot be readily obtained, approximate methods have been developed. In some cases solutions have been obtained by using experimental methods. As an example of this the photoelastic method of solving two-dimensional problems of elasticity may be mentioned. The photoelastic equipment may be found now at universities and also in many industrial research laboratories. The results of photoelastic experiments have proved especially useful in studying various cases of stress concentration at points of sharp variation of cross-sectional dimensions and at sharp fillets of reentrant corners. Without any doubt these results have considerably influenced the modern design of machine parts and helped in many cases to improve the construction by eliminating weak spots from which cracks may start.

Another example of the successful application of experiments in the solution of elasticity problems is the soap-film method for determining stresses in torsion and bending of prismatical bars. The difficult problems of the solution of partial differential equations with given boundary conditions are replaced in this case by measurements of slopes and deflections of a properly stretched and loaded soap film. The experiments show that in this way not only a visual picture of the stress distribution but also the necessary information regarding magnitude of stresses can be obtained with an accuracy sufficient for practical application.

Again, the electrical analogy which gives a means of investigating torsional stresses in shafts of variable diameter at the fillets and grooves is interesting. The analogy between the problem of bending of plates and the two-dimensional problem of elasticity has also been successfully applied in the solution of important engineering problems.

In the preparation of this book the intention was to give to engineers, in a simple form, the necessary fundamental knowledge of the theory of elasticity. It was also intended to bring together solutions of special problems which may be of practical importance and to describe approximate and experimental methods of the solution of elasticity problems.

Having in mind practical applications of the theory of elasticity, matters of more theoretical interest and those which have not at present any direct applications in engineering have been omitted in favor of the discussion of specific cases. Only by studying such cases with all the details and by comparing the results of exact investigations with the approximate solutions usually given in the elementary books on strength of materials can a designer acquire a thorough understanding of stress distribution in engineering structures, and learn to use, to his advantage, the more rigorous methods of stress analysis.

In the discussion of special problems in most cases the method of direct determination of stresses and the use of the compatibility equations in terms of stress components has been applied. This method is more familiar to engineers who are usually interested in the magnitude of stresses. By a suitable introduction of stress functions this method is also often simpler than that in which equations of equilibrium in terms of displacements are used.

In many cases the energy method of solution of elasticity problems has been used. In this way the integration of differential equations is replaced by the investigation of minimum conditions of certain integrals. Using Ritz's method this problem of variational calculus is reduced to a simple problem of finding a minimum of a function. In this manner useful approximate solutions can be obtained in many practically important cases.

To simplify the presentation, the book begins with the discussion of two-dimensional problems and only later, when the reader has familiarized himself with the various methods used in the solution of problems of the theory of elasticity, are three-dimensional problems discussed. The portions of the book that, although of practical importance, are such that they can be omitted during the first reading are put in small type. The reader may return to the study of such problems after finishing with the most essential portions of the book.

The mathematical derivations are put in an elementary form and usually do not require more mathematical knowledge than is given in engineering schools. In the cases of more complicated problems all necessary explanations and intermediate calculations are given so that the reader can follow without difficulty through all the derivations. Only in a few cases are final results given without complete derivations. Then the necessary references to the papers in which the derivations can be found are always given.

In numerous footnotes references to papers and books on the theory of elasticity which may be of practical importance are given. These references may be of interest to engineers who wish to study some special problems in more detail. They give also a picture of the modern development of the theory of elasticity and may be of some use to graduate students who are planning to take their work in this field.

In the preparation of the book the contents of a previous book ("Theory of Elasticity," vol. I, St. Petersburg, Russia, 1914) on the same subject, which represented a course of lectures on the theory of elasticity given in several Russian engineering schools, were used to a large extent.

The author was assisted in his work by Dr. L. H. Donnell and Dr. J. N. Goodier, who read over the complete manuscript and to whom he is indebted for many corrections and suggestions. The author takes this opportunity to thank also Prof. G. H. MacCullough, Dr. E. E. Weibel, Prof. M. Sadowsky, and Mr. D. H. Young, who assisted in the final preparation of the book by reading some portions of the manuscript. He is indebted also to Mr. L. S. Veenstra for the preparation of drawings and to Mrs. E. D. Webster for the typing of the manuscript.

S. TIMOSHENKO

UNIVERSITY OF MICHIGAN
December, 1933

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NOTATION

x, y, z	Rectangular coordinates.
r, θ	Polar coordinates.
ξ, η	Orthogonal curvilinear coordinates; sometimes rectangular coordinates.
R, ψ, θ	Spherical coordinates.
N	Outward normal to the surface of a body.
l, m, n	Direction cosines of the outward normal.
A	Cross-sectional area.
I_x, I_y	Moments of inertia of a cross section with respect to x- and y-axes.
I_p	Polar moment of inertia of a cross section.
g	Gravitational acceleration.
ρ	Density.
q	Intensity of a continuously distributed load.
p	Pressure.
X, Y, Z	Components of a body force per unit volume.
$\bar{X}, \bar{Y}, \bar{Z}$	Components of a distributed surface force per unit area.
M	Bending moment.
M_t	Torque.
$\sigma_x, \sigma_y, \sigma_z$	Normal components of stress parallel to x-, y-, and z-axes.
σ_n	Normal component of stress parallel to n.
σ_r, σ_θ	Radial and tangential normal stresses in polar coordinates.
σ_ξ, σ_η	Normal stress components in curvilinear coordinates.
$\sigma_r, \sigma_\theta, \sigma_z$	Normal stress components in cylindrical coordinates.
Θ	$= \sigma_x + \sigma_y + \sigma_z = \sigma_r + \sigma_\theta + \sigma_z$.
τ	Shearing stress.
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shearing-stress components in rectangular coordinates.
$\tau_{r\theta}$	Shearing stress in polar coordinates.
$\tau_{\xi\eta}$	Shearing stress in curvilinear coordinates.
$\tau_{r\theta}, \tau_{\theta z}, \tau_{rz}$	Shearing-stress components in cylindrical coordinates.
S	Total stress on a plane.
u, v, w	Components of displacements.
ϵ	Unit elongation.
$\epsilon_x, \epsilon_y, \epsilon_z$	Unit elongations in x-, y-, and z-directions.

$\epsilon_r, \epsilon_\theta$ Radial and tangential unit elongations in polar coordinates.

$e = \epsilon_x + \epsilon_y + \epsilon_z$ Volume expansion.

γ Unit shear.

$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ Shearing-strain components in rectangular coordinates.

$\gamma_{r\theta}, \gamma_{\theta z}, \gamma_{rz}$ Shearing-strain components in cylindrical coordinates.

E Modulus of elasticity in tension and compression.

G Modulus of elasticity in shear. Modulus of rigidity.

ν Poisson's ratio.

$\mu = G, \lambda = \dfrac{\nu E}{(1 + \nu)(1 - 2\nu)}$ Lamé's constants.

ϕ Stress function.

$\psi(z), \chi(z)$ Complex potentials; functions of the complex variable $z = x + iy$.

$\bar{z}$ The conjugate complex variable $x - iy$.

C Torsional rigidity.

θ Angle of twist per unit length.

$F = 2G\theta$ Used in torsional problems.

V Strain energy.

V_0 Strain energy per unit volume.

t Time.

T Certain interval of time. Temperature.

α Coefficient of thermal expansion.

CHAPTER 1

INTRODUCTION

1. Elasticity. All structural materials possess to a certain extent the property of *elasticity, i.e.*, if external forces, producing *deformation* of a structure, do not exceed a certain limit, the deformation disappears with the removal of the forces. Throughout this book it will be assumed that the bodies undergoing the action of external forces are *perfectly elastic, i.e.*, that they resume their initial form completely after removal of forces.

The molecular structure of elastic bodies will not be considered here. It will be assumed that the matter of an elastic body is *homogeneous* and continuously distributed over its volume so that the smallest element cut from the body possesses the same specific physical properties as the body. To simplify the discussion it will also be assumed that the body is *isotropic, i.e.*, that the elastic properties are the same in all directions.

Structural materials usually do not satisfy the above assumptions. Such an important material as steel, for instance, when studied with a microscope, is seen to consist of crystals of various kinds and various orientations. The material is very far from being homogeneous, but experience shows that solutions of the theory of elasticity based on the assumptions of homogeneity and isotropy can be applied to steel structures with very great accuracy. The explanation of this is that the crystals are very small; usually there are millions of them in one cubic inch of steel. While the elastic properties of a single crystal may be very different in different directions, the crystals are ordinarily distributed at random and the elastic properties of larger pieces of metal represent averages of properties of the crystals. So long as the geometrical dimensions defining the form of a body are large in comparison with the dimensions of a single crystal the assumption of homogeneity can be used with great accuracy, and if the crystals are orientated at random the material can be treated as isotropic.

When, due to certain technological processes such as rolling, a certain orientation of the crystals in a metal prevails, the elastic properties of the metal become different in different directions and the condition of *anistropy* must be considered. We have such a condition, for instance, in the case of cold-rolled copper.

2. Stress. Let Fig. 1 represent a body in equilibrium. Under the action of external forces $P_1, \ldots, P_7$, internal forces will be produced between the parts of the body. To study the magnitude of these forces at any point O, let us imagine the body divided into two parts A and B by a cross section mm through this point. Considering one of these parts, for instance, A, it can be stated that it is in equilibrium under the action of external forces $P_1, \ldots, P_7$ and the inner forces distributed over the cross section mm and representing the actions of the material of the part B on the material of the part A. It will be assumed that these forces are continuously distributed over the area mm in the same way that hydrostatic pressure or wind pressure is continuously distributed over the surface on which it acts. The magnitudes of such forces are usually defined by their *intensity*, *i.e.*, by the amount of force per unit area of the surface on which they act. In discussing internal forces this intensity is called *stress*.

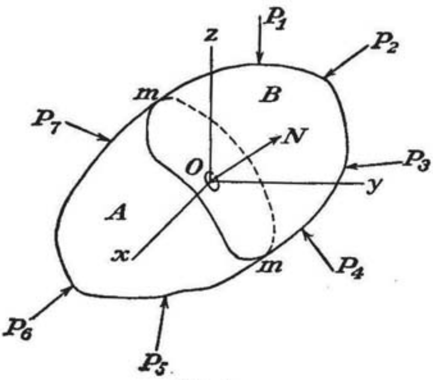


FIG. 1.

In the simplest case of a prismatical bar submitted to tension by forces uniformly distributed over the ends (Fig. 2), the internal forces are also uniformly distributed over any cross section mm. Hence the intensity of this distribution, *i.e.*, the stress, can be obtained by dividing the total tensile force P by the cross-sectional area A.

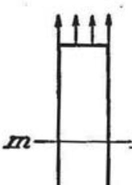


FIG. 2.

In the case just considered the stress was uniformly distributed over the cross section. In the general case of Fig. 1 the stress is not uniformly distributed over mm. To obtain the magnitude of stress acting on a small area δA, cut out from the cross section mm at any point O, we assume that the forces acting across this elemental area, due to the action of material of the part B on the material of the part A, can be reduced to a resultant δP. If we now continuously contract the elemental area δA, the limiting value of the ratio $\delta P/\delta A$ gives us the magnitude of the stress acting on the cross section mm at the point O. The limiting direction of the resultant δP is the direction of the stress. In the general case the direction of stress is inclined to the area δA on which it acts and we usually resolve it into two components: a *normal stress* perpendicular to the area, and a *shearing stress* acting in the plane of the area δA.

3. Notation for Forces and Stresses. There are two kinds of external forces which may act on bodies. Forces distributed over the surface of the body, such as the pressure of one body on another, or hydrostatic pressure, are called *surface forces.* Forces distributed over the volume of a body, such as gravitational forces, magnetic forces, or in the case of a body in motion, inertia forces, are called *body forces.* The surface force per unit area we shall usually resolve into three components parallel to the coordinate axes and use for these components the notation $\bar{X}, \bar{Y}, \bar{Z}$. We shall also resolve the body force per unit volume into three components and denote these components by X, Y, Z.

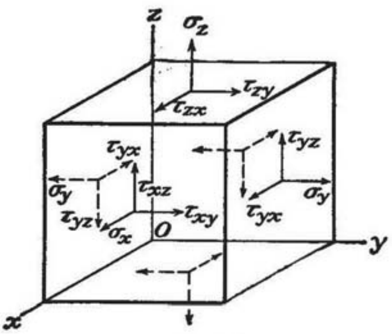


FIG. 3.

We shall use the letter σ for denoting normal stress and the letter τ for shearing stress. To indicate the direction of the plane on which the stress is acting, subscripts to these letters are used. If we take a very small cubic element at a point O, Fig. 1, with sides parallel to the coordinate axes, the notations for the components of stress acting on the sides of this element and the directions taken as positive are as indicated in Fig. 3. For the sides of the element perpendicular to the y-axis, for instance, the normal components of stress acting on these sides are denoted by σ_y. The subscript y indicates that the stress is acting on a plane normal to the y-axis. The normal stress is taken positive when it produces tension and negative when it produces compression.

The shearing stress is resolved into two components parallel to the coordinate axes. Two subscript letters are used in this case, the first indicating the direction of the normal to the plane under consideration and the second indicating the direction of the component of the stress. For instance, if we again consider the sides perpendicular to the y-axis, the component in the x-direction is denoted by τ_{yx} and that in the z-direction by τ_{yz}. The positive directions of the components of shearing stress on any side of the cubic element are taken as the positive directions of the coordinate axes if a tensile stress on the same side would have the positive direction of the corresponding axis. If the tensile stress has a direction opposite to the positive axis, the positive direction of the shearing-stress components should be reversed. Following this rule the positive directions of all the components of stress acting on the right side of the cubic element (Fig. 3) coincide with the positive directions of the coordinate axes. The positive directions are all reversed if we are considering the left side of this element.

4. Components of Stress. From the discussion of the previous article we see that, for each pair of parallel sides of a cubic element, such as in Fig. 3, one symbol is needed to denote the normal component of stress and two more symbols to denote the two components of shearing stress. To describe the stresses acting on the six sides of a cubic element three symbols, σ_x, σ_y, σ_z, are necessary for normal stresses; and six symbols, τ_{xy}, τ_{yx}, τ_{xz}, τ_{zx}, τ_{yz}, τ_{zy}, for shearing stresses. By a simple consideration of the equilibrium of the element the number of symbols for shearing stresses can be reduced to three.

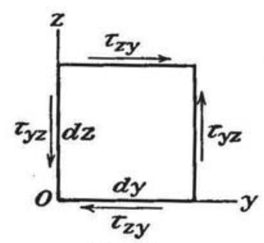


FIG. 4.

If we take the moments of the forces acting on the element about the x-axis, for instance, only the surface stresses shown in Fig. 4 need be considered. Body forces, such as the weight of the element, can be neglected in this instance, which follows from the fact that in reducing the dimensions of the element the body forces acting on it diminish as the cube of the linear dimensions while the surface forces diminish as the square of the linear dimensions. Hence, for a very small element, body forces are small quantities of higher order than surface forces and can be neglected in calculating the surface forces. Similarly, moments due to nonuniformity of distribution of normal forces are of higher order than those due to the shearing forces and vanish in the limit. Also the forces on each side can be considered to be the area of the side times the stress at the middle. Then denoting the dimensions of the small element in Fig. 4 by dx, dy, dz, the equation of equilibrium of this element, taking moments of forces about the x-axis, is

$$\tau_{zy}\, dx\, dy\, dz = \tau_{yz}\, dx\, dy\, dz$$

The two other equations can be obtained in the same manner. From these equations we find

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{zx} = \tau_{xz}, \qquad \tau_{zy} = \tau_{yz} \tag{1}$$

Hence for two perpendicular sides of a cubic element the components of shearing stress perpendicular to the line of intersection of these sides are equal.

The six quantities $\sigma_x, \sigma_y, \sigma_z, \tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}$ are therefore sufficient to describe the stresses acting on the coordinate planes through a point; these will be called the *components of stress* at the point.

It will be shown later (Art. 67) that with these six components the stress on any inclined plane through the same point can be determined.

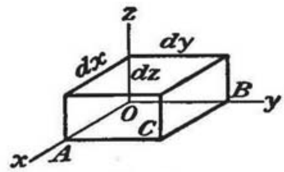


FIG. 5.

5. Components of Strain. In discussing the deformation of an elastic body it will be assumed that there are enough constraints to prevent the body from moving as a rigid body, so that no displacements of particles of the body are possible without a deformation of it.

In this book, only small deformations such as occur in engineering structures will be considered. The small displacements of particles of a deformed body will usually be resolved into components u, v, w parallel to the coordinate axes x, y, z, respectively. It will be assumed that these components are very small quantities varying continuously over the volume of the body. Consider a small element $dx\, dy\, dz$ of an elastic body (Fig. 5). If the body undergoes a deformation and u, v, w are the components of the displacement of the point O, the displacement in the x-direction of an adjacent point A on the x-axis is

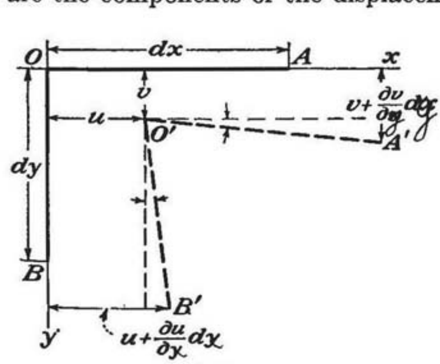


FIG. 6.

$$u + \frac{\partial u}{\partial x} dx$$

due to the increase $(\partial u/\partial x)\, dx$ of the function u with increase of the coordinate x. The increase in length of the element OA due to deformation is therefore $(\partial u/\partial x)\, dx$. Hence the *unit elongation* at point O in the x-direction is $\partial u/\partial x$. In the same manner it can be shown that the unit elongations in the y- and z-directions are given by the derivatives $\partial v/\partial y$ and $\partial w/\partial z$.

Let us consider now the distortion of the angle between the elements OA and OB, Fig. 6. If u and v are the displacements of the point O in the x- and y-directions, the displacement of the point A in the y-direction and of the point B in the x-direction are $v + (\partial v/\partial x)\,dx$ and $u + (\partial u/\partial y)\,dy$, respectively. Due to these displacements the new direction $O'A'$ of the element OA is inclined to the initial direction by the small angle indicated in the figure, equal to $\partial v/\partial x$. In the same manner the direction $O'B'$ is inclined to OB by the small angle $\partial u/\partial y$. From this it will be seen that the initially right angle AOB between the two elements OA and OB is diminished by the angle $\partial v/\partial x + \partial u/\partial y$. This is the *shearing strain* between the planes xz and yz. The shearing strains between the planes xy and xz and the planes yx and yz can be obtained in the same manner.

We shall use the letter ϵ for unit elongation and the letter γ for unit shearing strain. To indicate the directions of strain we shall use the same subscripts to these letters as for the stress components. Then from the above discussion

$$
\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x}, \qquad & \epsilon_y &= \frac{\partial v}{\partial y}, \qquad & \epsilon_z &= \frac{\partial w}{\partial z} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad & \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}
\end{aligned} \tag{2}
$$

It will be shown later that, having the three unit elongations in three perpendicular directions and three unit shear strains related to the same directions, the elongation in *any* direction and the distortion of the angle between *any* two directions can be calculated (see Art. 73). The six quantities $\epsilon_x, \ldots, \gamma_{yz}$ are called the *components of strain*.

6. Hooke's Law. The relations between the components of stress and the components of strain have been established experimentally and are known as *Hooke's law*. Imagine an elemental rectangular parallelopiped with the sides parallel to the coordinate axes and submitted to the action of normal stress σ_x uniformly distributed over two opposite sides. Experiments show that in the case of an isotropic material these normal stresses do not produce any distortion of angles of the element. The magnitude of the unit elongation of the element is given by the equation

$$\epsilon_x = \frac{\sigma_x}{E} \tag{a}$$

in which E is the *modulus of elasticity in tension*. Materials used in engineering structures have moduli which are very large in comparison with allowable stresses, and the unit elongation (a) is a very small quantity. In the case of structural steel, for instance, it is usually smaller than 0.001.

Extension of the element in the x-direction is accompanied by lateral contractions,

$$\epsilon_y = -\nu \frac{\sigma_x}{E}, \qquad \epsilon_z = -\nu \frac{\sigma_x}{E} \tag{b}$$

in which ν is a constant called *Poisson's ratio.* For many materials Poisson's ratio can be taken equal to 0.25. For structural steel it is usually taken equal to 0.3.

Equations (a) and (b) can be used also for simple compression. Within the elastic limit the modulus of elasticity and Poisson's ratio in compression are the same as in tension.

If the above element is submitted to the action of normal stresses $\sigma_x, \sigma_y, \sigma_z$, uniformly distributed over the sides, the resultant components of strain can be obtained by using Eqs. (a) and (b). Experiments show that to get these components we have to superpose the strain components produced by each of the three stresses. By this method of superposition we obtain the equations

$$\begin{aligned} \epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned} \tag{3}$$

In our further discussion we shall often use this *method of superposition* in calculating total deformations and stresses produced by several forces. This method is legitimate as long as the deformations are small and the corresponding small displacements do not affect substantially the action of the external forces. In such cases we neglect small changes in dimensions of deformed bodies and also small displacements of the points of application of external forces and base our calculations on initial dimensions and initial shape of the body. The resultant displacements will then be obtained by superposition in the form of linear functions of external forces, as in deriving Eqs. (3).

There are, however, exceptional cases in which small deformations cannot be neglected but must be taken into consideration. As an example of this kind the case of the simultaneous action on a thin bar of axial and lateral forces may be mentioned. Axial forces alone produce simple tension or compression, but they may have a substantial effect on the bending of the bar if they are acting simultaneously with lateral forces. In calculating the deformation of bars under such conditions, the effect of the deflection on the moment of the external forces must be considered, even though the deflections are very small.[1] Then the total deflection is no longer a linear function of the forces and cannot be obtained by simple superposition.

Equations (3) show that the relations between elongations and stresses are completely defined by two physical constants E and ν. The same constants can also be used to define the relation between shearing strain and shearing stress.

Let us consider the particular case of deformation of the rectangular parallelopiped in which $\sigma_y = -\sigma_z$ and $\sigma_x = 0$. Cutting out an element $abcd$ by planes parallel to the x-axis and at 45 deg. to the y- and z-axes (Fig. 7), it may be seen from Fig. 7b, by summing up the forces along and perpendicular to bc, that the normal stress on the sides of this element is zero and the shearing stress on the sides is

$$\tau = \tfrac{1}{2}(\sigma_z - \sigma_y) = \sigma_z \qquad (c)$$

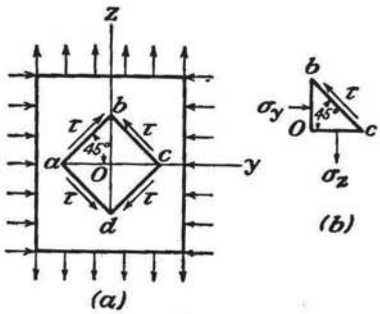


FIG. 7.

Such a condition of stress is called *pure shear*. The elongation of the vertical element Ob is equal to the shortening of the horizontal elements Oa and Oc, and neglecting a small quantity of the second order we conclude that the lengths ab and bc of the element do not change during deformation. The angle between the sides ab and bc changes, and the corresponding magnitude of shearing strain γ may be found from the triangle Obc. After deformation, we have

$$\frac{Oc}{Ob} = \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{1 + \epsilon_y}{1 + \epsilon_z}$$

Substituting, from Eqs. (3),

$$\epsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_y) = \frac{(1 + \nu)\sigma_z}{E}$$

$$\epsilon_y = -\frac{(1 + \nu)\sigma_z}{E}$$

and noting that for small γ

[1] Several examples of this kind can be found in S. Timoshenko, "Strength of Materials," vol. II, pp. 25–49.

$$\tan\left(\frac{\pi}{4}-\frac{\gamma}{2}\right)=\frac{\tan\frac{\pi}{4}-\tan\frac{\gamma}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\gamma}{2}}=\frac{1-\frac{\gamma}{2}}{1+\frac{\gamma}{2}}$$

we find

$$\gamma=\frac{2(1+\nu)\sigma_z}{E}=\frac{2(1+\nu)\tau}{E} \tag{4}$$

Thus the relation between shearing strain and shearing stress is defined by the constants E and ν. Often the notation

$$G=\frac{E}{2(1+\nu)} \tag{5}$$

is used. Then Eq. (4) becomes

$$\gamma=\frac{\tau}{G}$$

The constant G, defined by (5), is called the *modulus of elasticity in shear* or the *modulus of rigidity*.

If shearing stresses act on the sides of an element, as shown in Fig. 3, the distortion of the angle between any two coordinate axes depends only on shearing-stress components parallel to these axes and we obtain

$$\gamma_{xy}=\frac{1}{G}\tau_{xy},\qquad \gamma_{yz}=\frac{1}{G}\tau_{yz},\qquad \gamma_{xz}=\frac{1}{G}\tau_{xz} \tag{6}$$

The elongations (3) and the distortions (6) are independent of each other. Hence the general case of strain, produced by three normal and three shearing components of stress, can be obtained by superposition: on the three elongations given by Eqs. (3) are superposed three shearing strains given by Eqs. (6).

Equations (3) and (6) give the components of strain as functions of the components of stress. Sometimes the components of stress expressed as functions of the components of strain are needed. These can be obtained as follows. Adding equations (3) together and using the notations

$$\begin{aligned} e &= \epsilon_x+\epsilon_y+\epsilon_z \\ \Theta &= \sigma_x+\sigma_y+\sigma_z \end{aligned} \tag{7}$$

we obtain the following relation between the volume expansion e and the sum of normal stresses:

$$e=\frac{1-2\nu}{E}\Theta \tag{8}$$

In the case of a uniform hydrostatic pressure of the amount p we have

$$\sigma_x = \sigma_y = \sigma_z = -p$$

and Eq. (8) gives

$$e = -\frac{3(1-2\nu)p}{E}$$

which represents the relation between unit volume expansion e and hydrostatic pressure p.

The quantity $E/3(1-2\nu)$ is called the *modulus of volume expansion.*

Using notations (7) and solving Eqs. (3) for σ_x, σ_y, σ_z, we find

$$\begin{aligned} \sigma_x &= \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{1+\nu}\epsilon_x \\ \sigma_y &= \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{1+\nu}\epsilon_y \\ \sigma_z &= \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{1+\nu}\epsilon_z \end{aligned} \tag{9}$$

or using the notation

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{10}$$

and Eq. (5), these become

$$\begin{aligned} \sigma_x &= \lambda e + 2G\epsilon_x \\ \sigma_y &= \lambda e + 2G\epsilon_y \\ \sigma_z &= \lambda e + 2G\epsilon_z \end{aligned} \tag{11}$$

Problems

1. Show that Eqs. (1) continue to hold if the element of Fig. 4 is in motion and has an angular acceleration like a rigid body.

2. Suppose an elastic material contains a large number of evenly distributed small magnetized particles, so that a magnetic field exerts on any element $dx\,dy\,dz$ a moment $\mu\,dx\,dy\,dz$ about an axis parallel to the x-axis. What modification will be needed in Eqs. (1)?

3. Give some reasons why the formulas (2) will be valid for *small* strains only.

4. An elastic layer is sandwiched between two perfectly rigid plates, to which it is bonded. The layer is compressed between the plates, the compressive stress being σ_z. Supposing that the attachment to the plates prevents lateral strain ϵ_x, ϵ_y completely, find the apparent Young's modulus (*i.e.*, σ_z/ϵ_z) in terms of E and ν. Show that it is many times E if the material of the layer is nearly incompressible by hydrostatic pressure.

5. Prove that Eq. (8) follows from Eqs. (11), (10), and (5).

CHAPTER 2

PLANE STRESS AND PLANE STRAIN

7. Plane Stress. If a thin plate is loaded by forces applied at the boundary, parallel to the plane of the plate and distributed uniformly over the thickness (Fig. 8), the stress components σ_z, τ_{xz}, τ_{yz} are zero on both faces of the plate, and it may be assumed, tentatively, that they are zero also within the plate. The state of stress is then specified by σ_x, σ_y, τ_{xy} only, and is called *plane stress*. It may also be assumed that

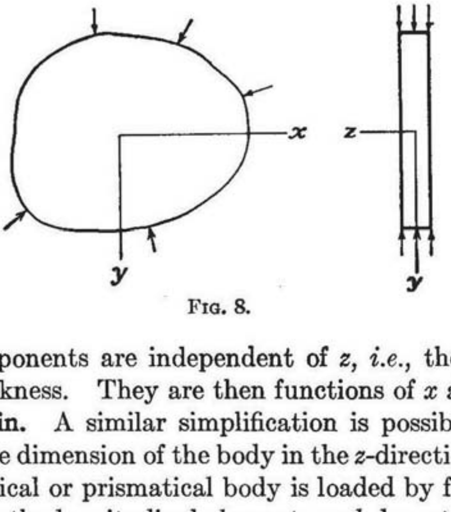


Fig. 8.

these three components are independent of z, *i.e.*, they do not vary through the thickness. They are then functions of x and y only.

8. Plane Strain. A similar simplification is possible at the other extreme when the dimension of the body in the z-direction is very large. If a long cylindrical or prismatical body is loaded by forces which are perpendicular to the longitudinal elements and do not vary along the length, it may be assumed that all cross sections are in the same condition. It is simplest to suppose at first that the end sections are confined between fixed smooth rigid planes, so that displacement in the axial direction is prevented. The effect of removing these will be examined later. Since there is no axial displacement at the ends, and, by symmetry, at the mid-section, it may be assumed that the same holds at every cross section.

There are many important problems of this kind—a retaining wall with lateral pressure (Fig. 9), a culvert or tunnel (Fig. 10), a cylindrical tube with internal pressure, a cylindrical roller compressed by forces in a diametral plane as in a roller bearing (Fig. 11). In each case of course the loading must not vary along the length. Since conditions are the same at all cross sections, it is sufficient to consider only a slice between two sections unit distance apart. The components u and v of the displacement are functions of x and y but are independent of the

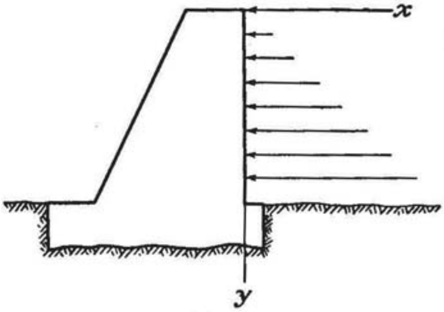


FIG. 9.

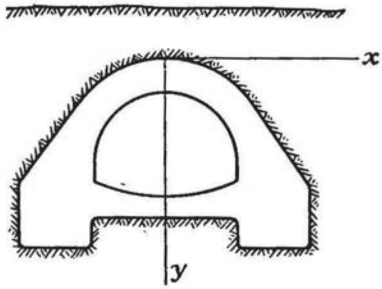


FIG. 10.

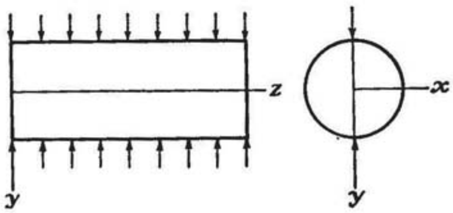


FIG. 11.

longitudinal coordinate z. Since the longitudinal displacement w is zero, Eqs. (2) give

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \qquad (a)$$

$$\epsilon_z = \frac{\partial w}{\partial z} = 0$$

The longitudinal normal stress σ_z can be found in terms of σ_x and σ_y by means of Hooke's law, Eqs. (3). Since $\epsilon_z = 0$ we find

$$\sigma_z - \nu(\sigma_x + \sigma_y) = 0$$

or

$$\sigma_z = \nu(\sigma_x + \sigma_y) \qquad (b)$$

These normal stresses act over the cross sections, including the ends, where they represent forces required to maintain the plane strain, and provided by the fixed smooth rigid planes.

By Eqs. (*a*) and (6), the stress components τ_{xz} and τ_{yz} are zero, and, by Eq. (*b*), σ_z can be found from σ_x and σ_y. Thus the plane strain problem, like the plane stress problem, reduces to the determination of σ_x, σ_y, and τ_{xy} as functions of x and y only.

9. Stress at a Point. Knowing the stress components σ_x, σ_y, τ_{xy} at any point of a plate in a condition of plane stress or plane strain, the stress acting on any plane through this point perpendicular to the plate and inclined to the x- and y-axes can be calculated from the equations of statics. Let O be a point of the stressed plate and suppose the stress components σ_x, σ_y, τ_{xy} are known (Fig. 12). To find the stress for any plane through the z-axis and inclined to the x- and y-axes, we take a plane BC parallel to it, at a small distance from O, so that this latter plane together with the coordinate planes cuts out from the plate a very small triangular prism OBC. Since the stresses vary continuously over the volume of the body the stress acting on the plane BC will approach the stress on the parallel plane through O as the element is made smaller.

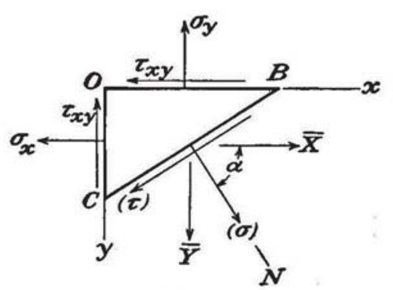


FIG. 12.

In discussing the conditions of equilibrium of the small triangular prism, the body force can be neglected as a small quantity of a higher order (page 4). Likewise, if the element is very small, we can neglect the variation of the stresses over the sides and assume that the stresses are uniformly distributed. The forces acting on the triangular prism can therefore be determined by multiplying the stress components by the areas of the sides. Let N be the direction of the normal to the plane BC, and denote the cosines of the angles between the normal N and the axes x and y by

$$\cos Nx = l, \qquad \cos Ny = m$$

Then, if A denotes the area of the side BC of the element, the areas of the other two sides are Al and Am.

If we denote by $\bar{X}$ and $\bar{Y}$ the components of stress acting on the side BC, the equations of equilibrium of the prismatical element give

$$\begin{aligned} \bar{X} &= l\sigma_x + m\tau_{xy} \\ \bar{Y} &= m\sigma_y + l\tau_{xy} \end{aligned} \qquad (12)$$

Thus the components of stress on any plane defined by the direction cosines l and m can easily be calculated from Eqs. (12), provided the three components of stress σ_x, σ_y, τ_{xy} at the point O are known.

Letting α be the angle between the normal N and the x-axis, so that $l = \cos\alpha$ and $m = \sin\alpha$, the normal and shearing components of stress on the plane BC are (from Eqs. 12)

$$\begin{aligned} \sigma &= \bar{X}\cos\alpha + \bar{Y}\sin\alpha = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha \\ &\qquad + 2\tau_{xy}\sin\alpha\cos\alpha \\ \tau &= \bar{Y}\cos\alpha - \bar{X}\sin\alpha = \tau_{xy}(\cos^2\alpha - \sin^2\alpha) \\ &\qquad + (\sigma_y - \sigma_x)\sin\alpha\cos\alpha \end{aligned} \tag{13}$$

It may be seen that the angle α can be chosen in such a manner that the shearing stress τ becomes equal to zero. For this case we have

$$\tau_{xy}(\cos^2\alpha - \sin^2\alpha) + (\sigma_y - \sigma_x)\sin\alpha\cos\alpha = 0$$

or

$$\frac{\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{1}{2}\tan 2\alpha \tag{14}$$

From this equation two perpendicular directions can be found for which the shearing stress is zero. These directions are called *principal directions* and the corresponding normal stresses *principal stresses.*

If the principal directions are taken as the x- and y-axes, τ_{xy} is zero and Eqs. (13) are simplified to

$$\begin{aligned} \sigma &= \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha \\ \tau &= \tfrac{1}{2}\sin 2\alpha(\sigma_y - \sigma_x) \end{aligned} \tag{13'}$$

The variation of the stress components σ and τ, as we vary the angle α, can be easily represented graphically by making a diagram in which σ and τ are taken as coordinates.[1] For each plane there will correspond a point on this diagram, the coordinates of which represent the values of σ and τ for this plane. Figure 13 represents such a diagram. For the planes perpendicular to the principal directions we obtain points A and B with abscissas σ_x and σ_y, respectively. Now it can be proved that the stress components for any plane BC with an angle α (Fig. 12) will be represented by coordinates of a point on the circle having AB as a diameter. To find this point it is only necessary to measure from the point A in the same direction as α is measured in Fig. 12 an arc subtending an angle equal to 2α. If D is the point obtained in this manner, then, from the figure,

$$OF = OC + CF = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha$$
$$DF = CD\sin 2\alpha = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\alpha$$

[1] This graphical method is due to O. Mohr, *Zivilingenieur,* 1882, p. 113. See also his "Technische Mechanik," 2d ed., 1914.

Comparing with Eqs. (13′) it is seen that the coordinates of point D give the numerical values of stress components on the plane BC at the angle α. To bring into coincidence the sign of the shearing component we take τ positive in the upward direction (Fig. 13) and consider shearing stresses as positive when they give a couple in the clockwise direction, as on the sides *bc* and *ad* of the element *abcd* (Fig. 13*b*). Shearing stresses of opposite direction, as on the sides *ab* and *dc* of the element, are considered as negative.[1]

As the plane BC rotates about an axis perpendicular to the xy-plane (Fig. 12) in the clockwise direction, and α varies from 0 to $\pi/2$, the

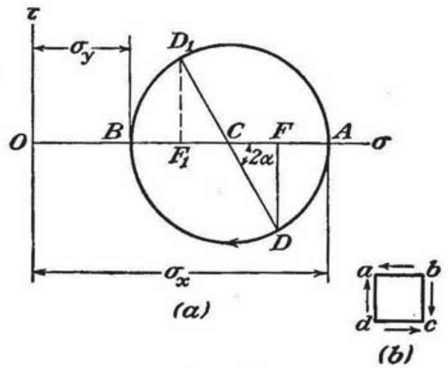


Fig. 13.

point D in Fig. 13 moves from A to B, so that the lower half circle determines the stress variation for all values of α within these limits. The upper half of the circle gives stresses for $\pi/2 \leqslant \alpha \leqslant \pi$.

Prolonging the radius CD to the point D_1 (Fig. 13), *i.e.*, taking the angle $\pi + 2\alpha$, instead of 2α, the stresses on the plane perpendicular to BC (Fig. 12) are obtained. This shows that the shearing stresses on two perpendicular planes are numerically equal as previously proved. As for normal stresses, we see from the figure that $OF_1 + OF = 2OC$, *i.e.*, the sum of the normal stresses over two perpendicular cross sections remains constant when the angle α changes.

The maximum shearing stress is given in the diagram (Fig. 13) by the maximum ordinate of the circle, *i.e.*, is equal to the radius of the circle. Hence

$$\tau_{\text{max.}} = \frac{\sigma_x - \sigma_y}{2} \tag{15}$$

It acts on the plane for which $\alpha = \pi/4$, *i.e.*, on the plane bisecting the angle between the two principal stresses.

[1] This rule is used only in the construction of Mohr's circle. Otherwise the rule given on p. 3 holds.

The diagram can be used also in the case when one or both principal stresses are negative (compression). It is only necessary to change the sign of the abscissa for compressive stress. In this manner Fig. 14*a* represents the case when both principal stresses are negative and Fig. 14*b* the case of pure shear.

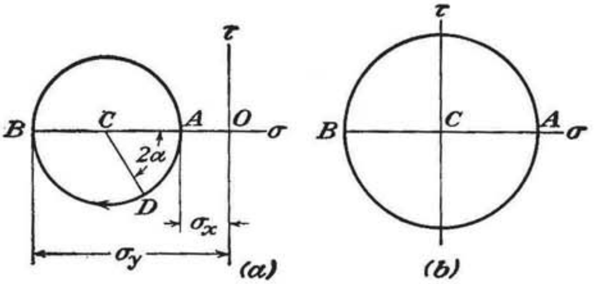


FIG. 14.

From Figs. 13 and 14 it is seen that the stress at a point can be resolved into two parts: One, uniform tension or compression, the magnitude of which is given by the abscissa of the center of the circle; and the other, pure shear, the magnitude of which is given by the radius of the circle. When several plane stress distributions are superposed, the uniform tensions or compressions can be added together

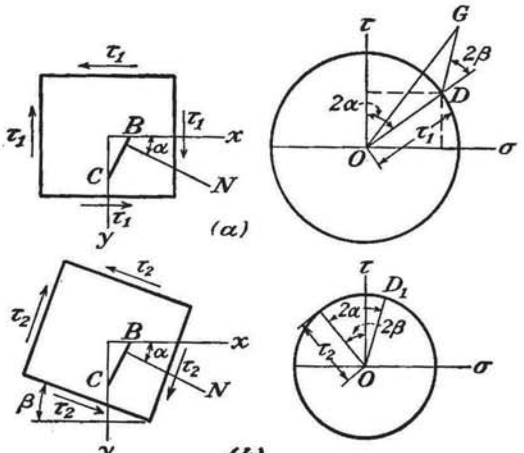


FIG. 15.

algebraically. The pure shears must be added together by taking into account the directions of the planes on which they are acting. It can be shown that, if we superpose two systems of pure shear whose planes of maximum shear make an angle of β with each other, the resulting system will be another case of pure shear. For example, Fig. 15 represents the determination of stress on any plane defined by α, produced by two pure shears of magnitude τ_1 and τ_2 acting one on the planes

xz and yz (Fig. 15a) and the other on the planes inclined to xz and yz by the angle β (Fig. 15b). In Fig. 15a the coordinates of point D represent the shear and normal stress on plane CB produced by the first system, while the coordinate of D_1 (Fig. 15b) gives the stresses on this plane for the second system. Adding OD and OD_1 geometrically we obtain OG, the resultant stress on the plane due to both systems, the coordinates of G giving us the shear and normal stress. Note that the magnitude of OG does not depend upon α. Hence, as the result of the superposition of two shears, we obtain a Mohr circle for pure shear, the magnitude of which is given by OG, the planes of maximum shear being inclined to the xz and yz planes by an angle equal to half the angle GOD.

A diagram, such as shown in Fig. 13, can be used also for determining principal stresses if the stress components σ_x, σ_y, τ_{xy} for any two perpendicular planes (Fig. 12) are known. We begin in such a case with the plotting of the two points D and D_1, representing stress conditions on the two coordinate planes (Fig. 16). In this manner the diameter DD_1 of the circle is obtained. Constructing the circle, the principal stresses σ_1 and σ_2 are obtained from the intersection of the circle with the abscissa axis. From the figure we find

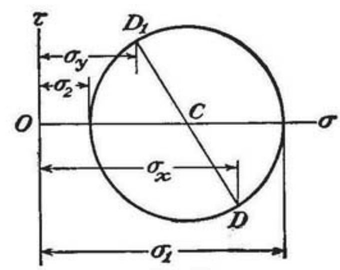


FIG. 16.

$$
\begin{aligned}
\sigma_1 &= OC + CD = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\
\sigma_2 &= OC - CD = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
\end{aligned} \tag{16}
$$

The maximum shearing stress is given by the radius of the circle, *i.e.*,

$$
\tau_{\max.} = \frac{1}{2}(\sigma_1 - \sigma_2) = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{17}
$$

In this manner all necessary features of the stress distribution at a point can be obtained if only the three stress components σ_x, σ_y, τ_{xy} are known.

10. Strain at a Point. When the strain components ϵ_x, ϵ_y, γ_{xy} at a point are known, the unit elongation for any direction, and the decrease of a right angle—the shearing strain—of any orientation at the point can be found. A line element PQ (Fig. 17a) between the points (x,y), $(x + dx,\ y + dy)$ is translated, stretched (or contracted) and rotated into the line element $P'Q'$ when the deformation occurs. The displacement components of P are u, v, and those of Q are

$$u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \qquad v + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

If $P'Q'$ in Fig. 17a is now translated so that P' is brought back to P, it is in the position PQ'' of Fig. 17b, and QR, RQ'' represent the components of the displacement of Q relative to P. Thus

$$QR = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \qquad RQ'' = \frac{\partial v}{\partial x} dx + \frac{\partial v}{dy} \partial y \tag{a}$$

The components of this relative displacement QS, SQ'', normal to PQ'' and along PQ'', can be found from these as

$$QS = -QR \sin\theta + RQ'' \cos\theta, \qquad SQ'' = QR \cos\theta + RQ'' \sin\theta \tag{b}$$

ignoring the small angle QPS in comparison with θ. Since the short line QS may be identified with an arc of a circle with center P, SQ''

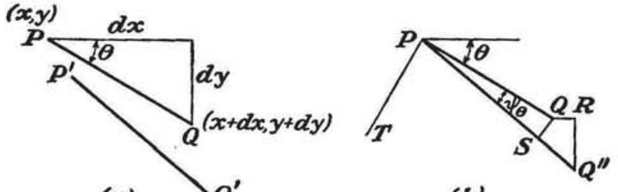


FIG. 17.

gives the stretch of PQ. The unit elongation of $P'Q'$, denoted by ϵ_θ, is SQ''/PQ. Using (b) and (a) we have

$$\begin{aligned}\epsilon_\theta &= \cos\theta\left(\frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}\right) + \sin\theta\left(\frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds}\right) \\ &= \frac{\partial u}{\partial x}\cos^2\theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta + \frac{\partial v}{\partial y}\sin^2\theta\end{aligned}$$

or

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \gamma_{xy}\sin\theta\cos\theta + \epsilon_y \sin^2\theta \tag{c}$$

which gives the unit elongation for any direction θ.

The angle ψ_θ through which PQ is rotated is QS/PQ. Thus from (b) and (a),

$$\psi_\theta = -\sin\theta\left(\frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}\right) + \cos\theta\left(\frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds}\right)$$

or

$$\psi_\theta = \frac{\partial v}{\partial x}\cos^2\theta + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\theta\cos\theta - \frac{\partial u}{\partial y}\sin^2\theta \tag{d}$$

The line element PT at right angles to PQ makes an angle $\theta + (\pi/2)$ with the x-direction, and its rotation $\psi_{\theta+\frac{\pi}{2}}$ is therefore given by (d) when $\theta + (\pi/2)$ is substituted for θ. Since $\cos[\theta + (\pi/2)] = -\sin\theta$, $\sin[\theta + (\pi/2)] = \cos\theta$, we find

$$\psi_{\theta+\frac{\pi}{2}} = \frac{\partial v}{\partial x}\sin^2\theta - \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\theta\cos\theta - \frac{\partial u}{\partial y}\cos^2\theta \qquad (e)$$

The shear strain γ_θ for the directions PQ, PT is $\psi_\theta - \psi_{\theta+\frac{\pi}{2}}$, so

$$\gamma_\theta = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)(\cos^2\theta - \sin^2\theta) + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)2\sin\theta\cos\theta$$

or

$$\tfrac{1}{2}\gamma_\theta = \tfrac{1}{2}\gamma_{xy}(\cos^2\theta - \sin^2\theta) + (\epsilon_y - \epsilon_x)\sin\theta\cos\theta \qquad (f)$$

Comparing (c) and (f) with (13), we observe that they may be obtained from (13) by replacing σ by ϵ_θ, τ by $\gamma_\theta/2$, σ_x by ϵ_x, σ_y by ϵ_y, τ_{xy} by $\gamma_{xy}/2$, and α by θ. Consequently for each deduction made from (13) as to σ and τ, there is a corresponding deduction from (c) and (f) as to ϵ_θ and $\gamma_\theta/2$. Thus there are two values of θ, differing by 90 deg., for which γ_θ is zero. They are given by

$$\frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = \tan 2\theta$$

The corresponding strains ϵ_θ are *principal strains*. A Mohr circle diagram analogous to Fig. 13 or Fig. 16 may be drawn, the ordinates representing $\gamma_\theta/2$ and the abscissas ϵ_θ. The principal strains ϵ_1, ϵ_2 will be the algebraically greatest and least values of ϵ_θ as a function of θ. The greatest value of $\gamma_\theta/2$ will be represented by the radius of the circle. Thus the greatest shearing strain $\gamma_{\theta\,\text{max.}}$ is given by

$$\gamma_{\theta\,\text{max.}} = \epsilon_1 - \epsilon_2$$

11. Measurement of Surface Strains. The strains, or unit elongations, on a surface are usually most conveniently measured by means of electric-resistance strain gauges.[1] The simplest form of such a gauge is a short length of wire insulated from and glued to the surface. When stretching occurs the resistance of the wire is increased, and the strain can thus be measured electrically. The effect is usually magnified by looping the wires backward and forward several times, to form several gauge lengths connected in series. The wire is glued between two tabs of paper, and the assembly glued to the surface.

The use of these gauges is simple when the principal directions are

[1] A detailed account of this method is given in the "Handbook of Experimental Stress Analysis," Chaps. 5 and 9.

known. One gauge is placed along each principal direction and direct measurements of ϵ_1, ϵ_2 obtained. The principal stresses σ_1, σ_2 may then be calculated from Hooke's law, Eqs. (3), with $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$, $\sigma_z = 0$, the last holding on the assumption that there is no stress acting on the surface to which the gauges are attached. Then

$$(1 - \nu^2)\sigma_1 = E(\epsilon_1 + \nu\epsilon_2), \qquad (1 - \nu^2)\sigma_2 = E(\epsilon_2 + \nu\epsilon_1)$$

When the principal directions are not known in advance, three measurements are needed. Thus the state of strain is completely determined if ϵ_x, ϵ_y, γ_{xy} can be measured. But since the strain gauges meas-

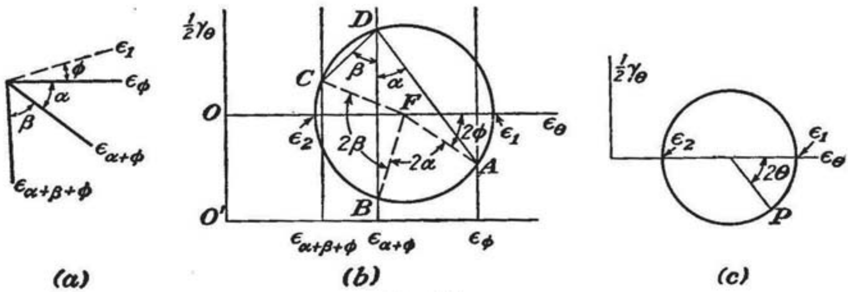


FIG. 18.

ure extensions, and not shearing strain directly, it is convenient to measure the unit elongations in three directions at the point. Such a set of gauges is called a "strain rosette." The Mohr circle can be drawn by the simple construction[1] given in Art. 12, and the principal strains can then be read off. The three gauges are represented by the three full lines in Fig. 18*a*. The broken line represents the (unknown) direction of the larger principal strain ϵ_1, from which the direction of the first gauge is obtained by a clockwise rotation ϕ.

If the x- and y-directions for Eqs. (*c*) and (*f*) of Art. 10 had been taken as the principal directions, ϵ_x would be ϵ_1, ϵ_y would be ϵ_2, and γ_{xy} would be zero. The equations would then be

$$\epsilon_\theta = \epsilon_1 \cos^2\theta + \epsilon_2 \sin^2\theta, \qquad \tfrac{1}{2}\gamma_\theta = -(\epsilon_1 - \epsilon_2)\sin\theta\cos\theta$$

where θ is the angle measured from the direction of ϵ_1. These may be written

$$\epsilon_\theta = \tfrac{1}{2}(\epsilon_1 + \epsilon_2) + \tfrac{1}{2}(\epsilon_1 - \epsilon_2)\cos 2\theta, \qquad \tfrac{1}{2}\gamma_\theta = -\tfrac{1}{2}(\epsilon_1 - \epsilon_2)\sin 2\theta$$

and these values are represented by the point P on the circle in Fig. 18*c*. If θ takes the value ϕ, P corresponds to the point A on the circle in Fig.

[1] Glenn Murphy, *J. Applied Mechanics* (*Trans. A.S.M.E.*), vol. 12, p. A-209, 1945; N. J. Hoff, *ibid.*

18*b*, the angular displacement from the ϵ_θ-axis being 2ϕ. The abscissa of this point is ϵ_ϕ, which is known. If θ takes the value $\phi + \alpha$, P moves to B, through a further angle $AFB = 2\alpha$, and the abscissa is the known value $\epsilon_{\alpha+\phi}$. If θ takes the value $\phi + \alpha + \beta$, P moves on to C, through a further angle $BFC = 2\beta$, and the abscissa is $\epsilon_{\alpha+\beta+\phi}$.

The problem is to draw the circle when these three abscissas and the two angles α, β are known.

12. Construction of Mohr Strain Circle for Strain Rosette. A temporary horizontal ϵ-axis is drawn horizontally from any origin O', Fig. 18*b*, and the three measured strains ϵ_ϕ, $\epsilon_{\alpha+\phi}$, $\epsilon_{\alpha+\beta+\phi}$ laid off along it. Verticals are drawn through these points. Selecting any point D on the vertical through $\epsilon_{\alpha+\phi}$, lines DA, DC are drawn at angles α and β to the vertical at D as shown, to meet the other two verticals at A and C. The circle drawn through D, A, and C is the required circle. Its center F is determined by the intersection of the perpendicular bisectors of CD, DA. The points representing the three gauge directions are A, B, and C. The angle AFB, being twice the angle ADB at the circumference, is 2α, and BFC is 2β. Thus A, B, C are at the required angular intervals round the circle, and have the required abscissas. The ϵ_θ axis can now be drawn as OF, and the distances from O to the intersections with the circle give ϵ_1, ϵ_2. The angle 2ϕ is the angle of FA below this axis.

13. Differential Equations of Equilibrium. We now consider the equilibrium of a small rectangular block of edges h, k, and unity (Fig. 19). The stresses acting on the faces 1, 2, 3, 4, and their positive directions are indicated in the figure. On account of the variation of stress throughout the material, the value of, for instance, σ_x is not quite the same for face 1 as for face 3. The symbols σ_x, σ_y, τ_{xy} refer to the point x, y, the mid-point of the rectangle in Fig. 19. The values at the mid-points of the faces are denoted by $(\sigma_x)_1$, $(\sigma_x)_3$, etc. Since the faces are very small, the corresponding forces are obtained by multiplying these values by the areas of the faces on which they act.[1]

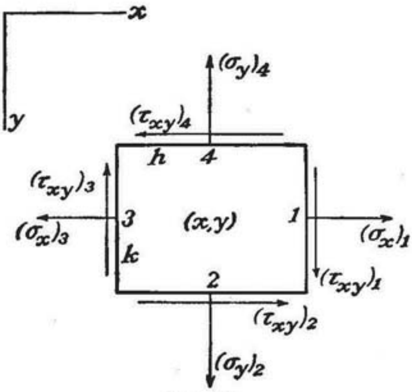


FIG. 19.

[1] More precise considerations would introduce terms of higher order which vanish in the final limiting process.

The body force on the block, which was neglected as a small quantity of higher order in considering the equilibrium of the triangular prism of Fig. 12, must be taken into consideration, because it is of the same order of magnitude as the terms due to the variations of the stress components which are now under consideration. If X, Y denote the components of body force per unit volume, the equation of equilibrium for forces in the x-direction is

$$(\sigma_x)_1 k - (\sigma_x)_3 k + (\tau_{xy})_2 h - (\tau_{xy})_4 h + Xhk = 0$$

or, dividing by hk,

$$\frac{(\sigma_x)_1 - (\sigma_x)_3}{h} + \frac{(\tau_{xy})_2 - (\tau_{xy})_4}{k} + X = 0$$

If now the block is taken smaller and smaller, *i.e.*, $h \to 0$, $k \to 0$, the limit of $[(\sigma_x)_1 - (\sigma_x)_3]/h$ is $\partial\sigma_x/\partial x$ by the definition of such a derivative. Similarly $[(\tau_{xy})_2 - (\tau_{xy})_4]/k$ becomes $\partial\tau_{xy}/\partial y$. The equation of equilibrium for forces in the y-direction is obtained in the same manner. Thus

$$\begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + Y &= 0 \end{aligned} \tag{18}$$

In practical applications the weight of the body is usually the only body force. Then, taking the y-axis downward and denoting by ρ the mass per unit volume of the body, Eqs. (18) become

$$\begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 0 \\ \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \rho g &= 0 \end{aligned} \tag{19}$$

These are the differential equations of equilibrium for two-dimensional problems.

14. Boundary Conditions. Equations (18) or (19) must be satisfied at all points throughout the volume of the body. The stress components vary over the volume of the plate, and when we arrive at the boundary they must be such as to be in equilibrium with the external forces on the boundary of the plate, so that external forces may be regarded as a continuation of the internal stress distribution. These conditions of equilibrium at the boundary can be obtained from Eqs. (12). Taking the small triangular prism OBC (Fig. 12), so that the side BC coincides with the boundary of the plate, as shown in Fig. 20, and denoting by $\bar{X}$ and $\bar{Y}$ the components of the surface forces per unit area at this point of the boundary, we have

$$\begin{aligned}\bar{X} &= l\sigma_x + m\tau_{xy}\\ \bar{Y} &= m\sigma_y + l\tau_{xy}\end{aligned} \tag{20}$$

in which l and m are the direction cosines of the normal N to the boundary.

In the particular case of a rectangular plate the coordinate axes are usually taken parallel to the sides of the plate and the boundary conditions (20) can be simplified. Taking, for instance, a side of the plate parallel to the x-axis we have for this part of the boundary the normal N parallel to the y-axis; hence $l = 0$ and $m = \pm 1$. Equations (20) then become

$$\bar{X} = \pm\tau_{xy}, \qquad \bar{Y} = \pm\sigma_y$$

Here the positive sign should be taken if the normal N has the positive direction of the y-axis and the negative sign for the opposite direction of N. It is seen from this that at the boundary the stress components become equal to the components of the surface forces per unit area of the boundary.

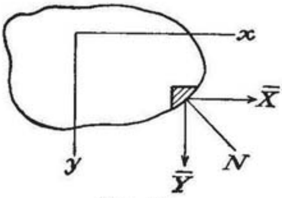


FIG. 20.

15. Compatibility Equations. The problem of the theory of elasticity usually is to determine the state of stress in a body submitted to the action of given forces. In the case of a two-dimensional problem it is necessary to solve the differential equations of equilibrium (18), and the solution must be such as to satisfy the boundary conditions (20). These equations, derived by application of the equations of statics for absolutely rigid bodies, and containing three stress components σ_x, σ_y, τ_{xy}, are not sufficient for the determination of these components. The problem is a statically indeterminate one, and in order to obtain the solution the elastic deformation of the body must also be considered.

The mathematical formulation of the condition for compatibility of stress distribution with the existence of continuous functions u, v, w defining the deformation will be obtained from Eqs. (2). In the case of two-dimensional problems only three strain components need be considered, namely,

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{a}$$

These three strain components are expressed by two functions u and v; hence they cannot be taken arbitrarily, and there exists a certain relation between the strain components which can easily be obtained from (a). Differentiating the first of the Eqs. (a) twice with respect to y, the second twice with respect to x, and the third once with respect to x and once with respect to y, we find

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \, \partial y} \tag{21}$$

This differential relation, called the *condition of compatibility*, must be satisfied by the strain components to secure the existence of functions u and v connected with the strain components by Eqs. (a). By using Hooke's law, [Eqs. (3)], the condition (21) can be transformed into a relation between the components of stress.

In the case of plane stress distribution (Art. 7), Eqs. (3) reduce to

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \qquad \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{22}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \tag{23}$$

Substituting in Eq. (21), we find

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \, \partial y} \tag{b}$$

This equation can be written in a different form by using the equations of equilibrium. For the case when the weight of the body is the only body force, differentiating the first of Eqs. (19) with respect to x and the second with respect to y and adding them, we find

$$2\frac{\partial^2 \tau_{xy}}{\partial x \, \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$$

Substituting in Eq. (b), the compatibility equation in terms of stress components becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \tag{24}$$

Proceeding in the same manner with the general equations of equilibrium (18) we find

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \tag{25}$$

In the case of plane strain (Art. 8), we have

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

and from Hooke's law (Eqs. 3), we find

$$\begin{aligned}\epsilon_x &= \frac{1}{E}[(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y] \\ \epsilon_y &= \frac{1}{E}[(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x]\end{aligned} \tag{26}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \tag{27}$$

Substituting in Eq. (21), and using, as before, the equations of equilibrium (19), we find that the compatibility equation (24) holds also for plane strain. For the general case of body forces we obtain from Eqs. (21) and (18) the compatibility equation in the following form:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \tag{28}$$

The equations of equilibrium (18) or (19) together with the boundary conditions (20) and one of the above compatibility equations give us a system of equations which is usually sufficient for the complete determination of the stress distribution in a two-dimensional problem.[1] The particular cases in which certain additional considerations are necessary will be discussed later (page 117). It is interesting to note that in the case of constant body forces the equations determining stress distribution do not contain the elastic constants of the material. Hence the stress distribution is the same for all isotropic materials, provided the equations are sufficient for the complete determination of the stresses. The conclusion is of practical importance: we shall see later that in the case of transparent materials, such as glass or xylonite, it is possible to determine stresses by an optical method using polarized light (page 131). From the above discussion it is evident that experimental results obtained with a transparent material in most cases can be applied immediately to any other material, such as steel.

It should be noted also that in the case of constant body forces the compatibility equation (24) holds both for the case of plane stress and for the case of plane strain. Hence the stress distribution is the same in these two cases, provided the shape of the boundary and the external forces are the same.[2]

[1] In plane stress there are compatibility conditions other than (21) which are in fact violated by our assumptions. It is shown in Art. 84 that in spite of this the method of the present chapter gives good approximations for thin plates.

[2] This statement may require modification when the plate or cylinder has holes, for then the problem can be correctly solved only by considering the displacements as well as the stresses. See Art. 39.

16. Stress Function. It has been shown that a solution of two-dimensional problems reduces to the integration of the differential equations of equilibrium together with the compatibility equation and the boundary conditions. If we begin with the case when the weight of the body is the only body force, the equations to be satisfied are (see Eqs. 19 and 24)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g = 0 \qquad (a)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \qquad (b)$$

To these equations the boundary conditions (20) should be added. The usual method of solving these equations is by introducing a new function, called the *stress function*.[1] As is easily checked, Eqs. (*a*) are satisfied by taking any function ϕ of x and y and putting the following expressions for the stress components:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - \rho g y, \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} - \rho g y, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y} \qquad (29)$$

In this manner we can get a variety of solutions of the equations of equilibrium (*a*). The true solution of the problem is that which satisfies also the compatibility equation (*b*). Substituting expressions (29) for the stress components into Eq. (*b*) we find that the stress function ϕ must satisfy the equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad (30)$$

Thus the solution of a two-dimensional problem, when the weight of the body is the only body force, reduces to finding a solution of Eq. (30) which satisfies the boundary conditions (20) of the problem. In the following chapters this method of solution will be applied to several examples of practical interest.

Let us now consider a more general case of body forces and assume that these forces have a potential. Then the components X and Y in Eqs. (18) are given by the equations

[1] This function was introduced in the solution of two-dimensional problems by G. B. Airy, *Brit. Assoc. Advancement Sci. Rept.*, 1862, and is sometimes called the *Airy stress function*.

$$X = -\frac{\partial V}{\partial x}$$
$$Y = -\frac{\partial V}{\partial y} \tag{c}$$

in which V is the potential function. Equations (18) become

$$\frac{\partial}{\partial x}(\sigma_x - V) + \frac{\partial \tau_{xy}}{\partial y} = 0$$
$$\frac{\partial}{\partial y}(\sigma_y - V) + \frac{\partial \tau_{xy}}{\partial x} = 0$$

These equations are of the same form as Eqs. (a) and can be satisfied by taking

$$\sigma_x - V = \frac{\partial^2 \phi}{\partial y^2}, \qquad \sigma_y - V = \frac{\partial^2 \phi}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \, \partial y} \tag{31}$$

in which ϕ is the stress function. Substituting expressions (31) in the compatibility equation (25) for plane stress distribution, we find

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -(1 - \nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \tag{32}$$

An analogous equation can be obtained for the case of plane strain.

When the body force is simply the weight, the potential V is $-\rho g y$. In this case the right-hand side of Eq. (32) reduces to zero. By taking the solution $\phi = 0$ of (32), or of (30), we find the stress distribution from (31), or (29),

$$\sigma_x = -\rho g y, \qquad \sigma_y = -\rho g y, \qquad \tau_{xy} = 0 \tag{d}$$

as a possible state of stress due to gravity. This is a state of hydrostatic pressure $\rho g y$ in two dimensions, with zero stress at $y = 0$. It can exist in a plate or cylinder of any shape provided the corresponding boundary forces are applied. Considering a boundary element as in Fig. 12, Eqs. (13) show that there must be a normal *pressure* $\rho g y$ on the boundary, and zero shear stress. If the plate or cylinder is to be supported in some other manner we have to superpose a boundary normal *tension* $\rho g y$ and the new supporting forces. The two together will be in equilibrium, and the determination of their effects is a problem of boundary forces only, without body forces.[1]

Problems

1. Show that Eqs. (12) remain valid when the element of Fig. 12 has acceleration.

2. Find graphically the principal strains and their directions from rosette measurements

$$\epsilon_\phi = 2 \times 10^{-3}, \qquad \epsilon_{\alpha+\phi} = 1.35 \times 10^{-3}, \qquad \epsilon_{\alpha+\beta+\phi} = 0.95 \times 10^{-3} \text{ in. per inch}$$

where $\alpha = \beta = 45°$.

[1] This problem, and the general case of a potential V such that the right-hand side of Eq. (32) vanishes, have been discussed by M. Biot, *J. Applied Mechanics* (*Trans. A.S.M.E.*), 1935, p. A-41.

3. Show that the line elements at the point x, y which have the maximum and minimum rotation are those in the two perpendicular directions θ determined by

$$\tan 2\theta = \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) \Big/ \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

4. The stresses in a rotating disk (of unit thickness) can be regarded as due to centrifugal force as body force in a stationary disk. Show that this body force is derivable from the potential $V = -\frac{1}{2}\rho\omega^2(x^2 + y^2)$, where ρ is the density, and ω the angular velocity of rotation (about the origin).

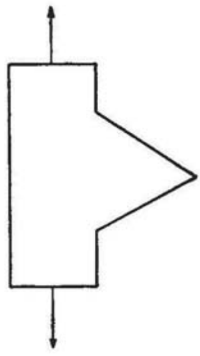
5. A disk with its axis horizontal has the gravity stress represented by Eqs. (*d*) of Art. 16. Make a sketch showing the boundary forces which support its weight. Show by another sketch the auxiliary problem of boundary forces which must be solved when the weight is entirely supported by the reaction of a horizontal surface on which the disk stands.

6. A cylinder with its axis horizontal has the gravity stress represented by Eqs. (*d*) of Art. 16. Its ends are confined between smooth fixed rigid planes which maintain the condition of *plane strain*. Sketch the forces acting on its surface, including the ends.

7. Using the stress-strain relations, and Eqs. (*a*) of Art. 15 in the equations of equilibrium (18), show that in the absence of body forces the displacements in problems of plane stress must satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1 + \nu}{1 - \nu}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

and a companion equation.



8. The figure represents a "tooth" on a plate in a state of plane stress in the plane of the paper. The faces of the tooth (the two straight lines) are free from force. Prove that there is no stress at all at the apex of the tooth. (*N.B.*: The same conclusion *cannot* be drawn for a reentrant, *i.e.*, internal, corner.)

CHAPTER 3

TWO-DIMENSIONAL PROBLEMS IN RECTANGULAR COORDINATES

17. Solution by Polynomials. It has been shown that the solution of two-dimensional problems, when body forces are absent or are constant, is reduced to the integration of the differential equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad (a)$$

having regard to boundary conditions (20). In the case of long rectangular strips, solutions of Eq. (a) in the form of polynomials are of interest. By taking polynomials of various degrees, and suitably adjusting their coefficients, a number of practically important problems can be solved.[1]

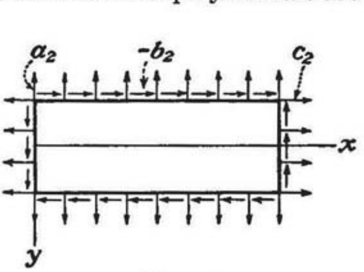


Fig. 21.

Beginning with a polynomial of the second degree

$$\phi_2 = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2 \qquad (b)$$

which evidently satisfies Eq. (a), we find from Eqs. (29), putting $\rho g = 0$,

$$\sigma_x = \frac{\partial^2 \phi_2}{\partial y^2} = c_2, \qquad \sigma_y = \frac{\partial^2 \phi_2}{\partial x^2} = a_2, \qquad \tau_{xy} = -\frac{\partial^2 \phi_2}{\partial x \, \partial y} = -b_2$$

All three stress components are constant throughout the body, *i.e.*, the stress function (b) represents a combination of uniform tensions or compressions[2] in two perpendicular directions and a uniform shear. The forces on the boundaries must equal the stresses at these points as discussed on page 23; in the case of a rectangular plate with sides parallel to the coordinate axes these forces are shown in Fig. 21.

[1] A. Mesnager, *Compt. rend.*, vol. 132, p. 1475, 1901. See also A. Timpe, *Z. Math. Physik*, vol. 52, p. 348, 1905.

[2] This depends on the sign of coefficients a_2 and b_2. The directions of stresses indicated in Fig. 21 are those corresponding to positive values of a_2, b_2, c_2.

Let us consider now a stress function in the form of a polynomial of the third degree:

$$\phi_3 = \frac{a_3}{3 \cdot 2} x^3 + \frac{b_3}{2} x^2 y + \frac{c_3}{2} x y^2 + \frac{d_3}{3 \cdot 2} y^3 \qquad (c)$$

This also satisfies Eq. (*a*). Using Eqs. (29) and putting $\rho g = 0$, we find

$$\sigma_x = \frac{\partial^2 \phi_3}{\partial y^2} = c_3 x + d_3 y$$

$$\sigma_y = \frac{\partial^2 \phi_3}{\partial x^2} = a_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2 \phi_3}{\partial x \, \partial y} = -b_3 x - c_3 y$$

For a rectangular plate, taken as in Fig. 22, assuming all coefficients except d_3 equal to zero, we obtain pure bending. If only coefficient a_3 is different from zero, we obtain pure bending by normal stresses applied to the sides $y = \pm c$ of the plate. If coefficient b_3 or c_3 is taken

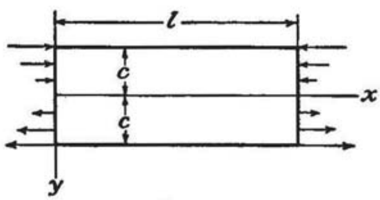


Fig. 22.

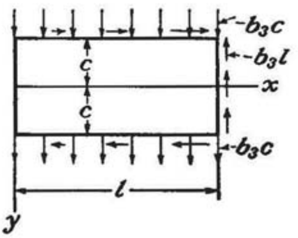


Fig. 23.

different from zero, we obtain not only normal but also shearing stresses acting on the sides of the plate. Figure 23 represents, for instance, the case in which all coefficients, except b_3 in function (*c*), are equal to zero. The directions of stresses indicated are for b_3 positive. Along the sides $y = \pm c$ we have uniformly distributed tensile and compressive stresses, respectively, and shearing stresses proportional to x. On the side $x = l$ we have only the constant shearing stress $-b_3 l$, and there are no stresses acting on the side $x = 0$. An analogous stress distribution is obtained if coefficient c_3 is taken different from zero.

In taking the stress function in the form of polynomials of the second and third degrees we are completely free in choosing the magnitudes of the coefficients, since Eq. (*a*) is satisfied whatever values they may have. In the case of polynomials of higher degrees Eq. (*a*) is satisfied only if certain relations between the coefficients are satisfied. Taking, for instance, the stress function in the form of a polynomial of the fourth degree,

$$\phi_4 = \frac{a_4}{4 \cdot 3} x^4 + \frac{b_4}{3 \cdot 2} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{3 \cdot 2} x y^3 + \frac{e_4}{4 \cdot 3} y^4 \qquad (d)$$

and substituting it into Eq. (a), we find that the equation is satisfied only if

$$e_4 = -(2c_4 + a_4)$$

The stress components in this case are

$$\begin{aligned}
\sigma_x &= \frac{\partial^2 \phi_4}{\partial y^2} = c_4 x^2 + d_4 xy - (2c_4 + a_4) y^2 \\
\sigma_y &= \frac{\partial^2 \phi_4}{\partial x^2} = a_4 x^2 + b_4 xy + c_4 y^2 \\
\tau_{xy} &= \frac{\partial^2 \phi_4}{\partial x \, \partial y} = -\frac{b_4}{2} x^2 - 2c_4 xy - \frac{d_4}{2} y^2
\end{aligned}$$

Coefficients $a_4, \ldots, d_4$ in these expressions are arbitrary, and by suitably adjusting them we obtain various conditions of loading of a rectangular plate. For instance, taking all coefficients except d_4 equal to zero, we find

$$\sigma_x = d_4 xy, \qquad \sigma_y = 0, \qquad \tau_{xy} = -\frac{d_4}{2} y^2 \qquad (e)$$

Assuming d_4 positive, the forces acting on the rectangular plate shown in Fig. 24 and producing the stresses (e) are as given. On the longitudinal sides $y = \pm c$ are uniformly distributed shearing forces; on the ends shearing forces are distributed according to a parabolic law. The shearing forces acting on the boundary of the plate reduce to the couple[1]

$$M = \frac{d_4 c^2 l}{2} \cdot 2c - \frac{1}{3} \frac{d_4 c^2}{2} \cdot 2c \cdot l = \frac{2}{3} d_4 c^3 l$$

This couple balances the couple produced by the normal forces along the side $x = l$ of the plate.

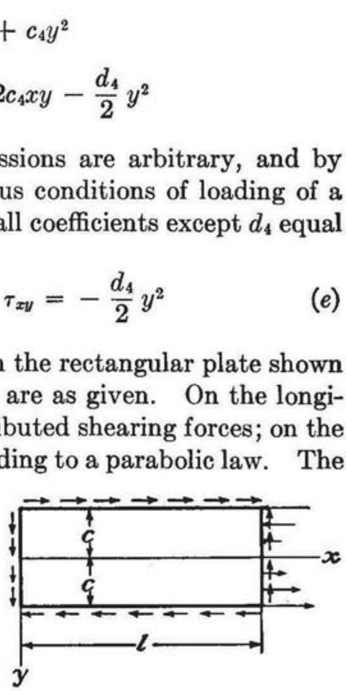


FIG. 24.

Let us consider a stress function in the form of a polynomial of the fifth degree.

$$\phi_5 = \frac{a_5}{5 \cdot 4} x^5 + \frac{b_5}{4 \cdot 3} x^4 y + \frac{c_5}{3 \cdot 2} x^3 y^2 + \frac{d_5}{3 \cdot 2} x^2 y^3 + \frac{e_5}{4 \cdot 3} x y^4 + \frac{f_5}{5 \cdot 4} y^5 \qquad (f)$$

[1] The thickness of the plate is taken equal to unity.